3.6

Closing Time
The Closure Property

LEARNING GOALS

In this lesson, you will:

- Compare functions that are closed under addition, subtraction, and multiplication to functions that are not closed under these operations.
- Analyze the meaning for polynomials to be closed under an operation.
- Compare integer and polynomial operations.

KEY TERM

- closed under an operation

The word "closure" can mean many things depending on the context.

- In business, closure is a process in which an organization can no longer operate. For instance, closure for a business may be caused by an organization going bankrupt.
- In psychology, closure is a person's emotional need for the conclusion of a difficult event in their life.
- In government, closure, which is also referred to as "cloture," is a procedure by which the Senate can vote to place a time limit on consideration of a bill.

Closure is also an important term in mathematics. Can you think of any other meanings for the word closure?

PROBLEM 1 Closed for Business

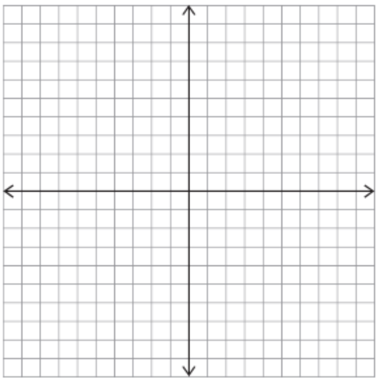
In this chapter you have learned the properties of polynomials in different representations.

Graphically, polynomials are:	Algebraically, polynomials are:	In a table of values, polynomials are:
• smooth • continuous • increase or decrease to infinity as x approaches positive or negative infinity	• written in the form $ax^n + bx^{n-1} + \cdots$	• made up of real numbers • increase or decrease to infinity as x approaches positive or negative infinity

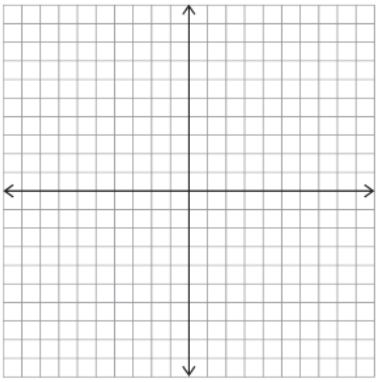
You have studied many different types of functions. A function has a unique output for every input value. However, a function does not necessarily have to be a polynomial function.

1. Sketch the graphs of two functions that are not polynomial functions. Explain your reasoning.

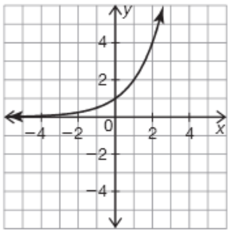
a.

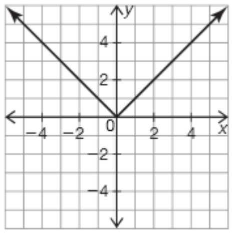


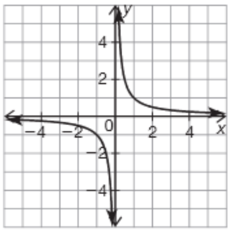
b.

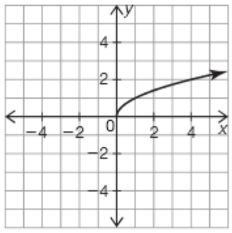


2. Analyze the graphs of the functions shown. Describe why each function is not a polynomial function.









Throughout this chapter you added, subtracted, or multiplied two or more polynomial functions to build a new polynomial function. You did this using a graph, algebra, and a table of values. When an operation is performed on any number or expression in a set and the result is in the same set, it is said to be **closed under that operation**. Are polynomials closed under addition, subtraction, and multiplication? In other words, when you add, subtract, or multiply polynomial functions, will you *always* create another polynomial function?

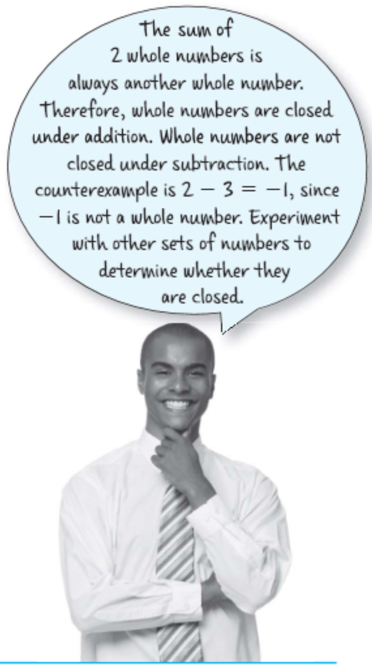
Before answering this question, let's analyze closure within the real number system.

3. Determine whether each set within the Real Number System is closed under addition, subtraction, multiplication, and division.

a. Complete the table. If a set is not closed under a given operation, provide a counterexample.

	Addition	Subtraction	Multiplication	Division
Natural Numbers {1, 2, 3, 4, . . .}				
Whole Numbers {0, 1, 2, 3, . . .}	Yes	No $2 - 3 = -1$		
Integers {. . . −2, −1, 0, 1, 2 . . .}				
Rational Can be represented as the ratio of two integers				
Irrational Cannot be represented as the ratio of two integers				

b. What patterns do you notice?



4. Determine whether polynomial functions are closed under addition, subtraction, multiplication, and division?

   a. Write 5 polynomials with various degrees that you will use to explore closure.

$y_1 =$ ______________	$y_2 =$ ______________
$y_3 =$ ______________	
$y_4 =$ ______________	$y_5 =$ ______________

   b. Determine whether the polynomials are closed under addition, subtraction, multiplication, and division. Show all work and explain your reasoning.

   c. How do you know when a polynomial is not closed under a given operation? Explain your reasoning in terms of the graph, table, and algebraic representation.

   d. Have you proven that polynomials are closed under a given operation? Have you proven that polynomials are not closed under a given operation? Explain your reasoning.

PROBLEM 2 Okay Then, Prove It!

In the previous problem, *Closed For Business*, you conjectured that integers and polynomials are both closed under addition, subtraction, and multiplication. You also determined through counterexamples that integers and polynomials are not closed under division.

1. Similarities between integer and polynomial operations are shown in the table.

	Integer Example	Polynomial Example
Addition	400 + 30 + 7 + 20 + 5 400 + 50 + 12	$4x^2 + 3x + 7$ $+ \quad 2x + 5$ $4x^2 + 5x + 12$
Subtraction	400 + 30 + 7 − (20 + 5) 400 + 10 + 2	$4x^2 + 3x + 7$ $- \quad (2x + 5)$ $4x^2 + x + 2$
Multiplication	400 + 30 + 7 × 20 + 5 2000 + 150 + 35 8000 + 600 + 140 8000 + 2600 + 290 + 35	$4x^2 + 3x + 7$ $\times \quad 2x + 5$ $20x^2 + 15x + 35$ $8x^3 + 6x^2 + 14x$ $8x^3 + 26x^2 + 29x + 35$
Division	$\frac{437}{25} = 17\ R12$	$\frac{4x^2 + 3x + 7}{2x + 5} = (2x - 3)\ R(-x + 22)$

a. Describe the similarities between polynomial and integer operations.

b. In what ways is the distributive property essential to performing operations with integers and polynomials?

c. How does this example demonstrate that polynomials are not closed under division?

d. Verify that the polynomial division was performed correctly.

You have explored operations under various polynomials. It appears as though polynomials are closed under addition, subtraction, and multiplication, but these examples do not constitute a proof. The real number system is closed, but discovering that polynomials are analogous to the real number system does not allow you to assume that polynomials are also closed. The worked example shows you how to formally prove that polynomials are closed under addition.

Consider the two polynomial functions $f(x)$ and $g(x)$.

$f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$

$g(x) = b_nx^n + b_{n-1}x^{n-1} + \cdots + b_1x + b_0$

You can show that the polynomials are closed under addition.

Step 1: Write the sum $f(x) + g(x)$. Because the polynomials have multiple terms, it is best to arrange the sum vertically.

$$\begin{array}{rcccccccc} & a_nx^n & + & a_{n-1}x^{n-1} & + \cdots + & a_1x & + & a_0 \\ + & b_nx^n & + & b_{n-1}x^{n-1} & + \cdots + & b_1x & + & b_0 \\ \hline \end{array}$$

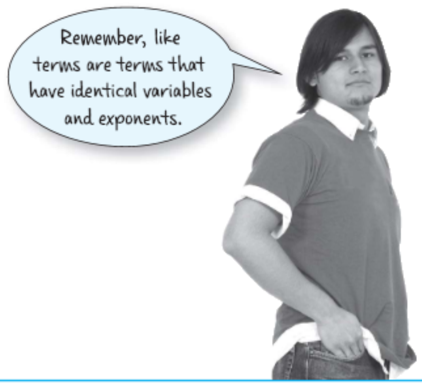
Step 2: Add the polynomials by combining like terms.

$$\begin{array}{rcccccccc} & a_nx^n & + & a_{n-1}x^{n-1} & + \cdots + & a_1x & + & a_0 \\ + & b_nx^n & + & b_{n-1}x^{n-1} & + \cdots + & b_1x & + & b_0 \\ \hline \end{array}$$

$$(a_n + b_n)x^n + (a_{n-1} + b_{n-1})x^{n-1} + \cdots + (a_1 + b_1)x + (a_0 + b_0)$$

Step 3: In the sum, each coefficient is of the form $a_n + b_n$. A coefficient $a_n + b_n$ is a real number because a_n and b_n are real numbers, and the real numbers are closed under addition.

Step 4: The sum of the polynomials $f(x)$ and $g(x)$ is in the form of a polynomial function with a real coefficient. Therefore, polynomials are closed under addition.



2. Consider the two polynomial functions $f(x)$ and $g(x)$.

$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$

$g(x) = b_n x^n + b_{n-1}x^{n-1} + \cdots + b_1 x + b_0$

a. Prove that polynomials are closed under subtraction.

3

b. Use the multiplication table to prove that polynomials are closed under multiplication.

$\bullet$	$a_n x^n$	$a_{n-1}x^{n-1}$	$\cdots$	$a_1 x$	a_0
$b_n x_n$					
$b_{n-1}x_{n-1}$					
$\vdots$					
$b_1 x$					
b_0					

Be prepared to share your solutions and methods.